IEEE

Philadelphia Section
Engineer's Week

Characterization of Rotating and Spinning Bodies with Quaternions

James K Beard, Ph.D.
Independent Consultant
http://www.JamesKBeard.com

Our Topics Tonight

- What quaternions are and why they are important
- Some places that quaternions are used
- Quaternion rotation explained and simplified
- Why quaternions are simpler than rotation matrices
- How quaternions are more accurate in computing
- Quaternions in Euler's equations of motion for rotating bodies
- Using quaternions in characterizing position and velocity
- Examples, with animations
- Selected references

What Are Quaternions?
Why Are They Useful?

- Quaternions are
  - A way of working with rotating rigid bodies
  - The “sum” of a scalar and a vector
- Why are quaternions important?
  - Their use takes the pain out of modeling aircraft, missiles, spinning bodies, etc.
  - They are easily incorporated into
    - Models that include rotating rigid bodies
    - Computer program for analysis or in embedded functions in systems
    - Inertial navigation units and autopilots

Where Do We Find Quaternions in Use?

- Your airplanes
  - The autopilot and INU keeps track of latitude, longitude, altitude
  - Quaternions are used in characterizing position
  - Aircraft orientation – roll, pitch and yaw
- Your quad drone – in its autopilot
- Your cell phone
- Your car
- In space
  - Launch vehicles
  - Spacecraft

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F-35A Maneuvers to Refuel from KC-135

By MSgt John Nimmo Sr. –
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What Are Quaternions, Exactly?

$$q = q_0 + \underline{v}_q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- Algebraically
  - A scalar associated with a vector in 3-space
  - Or, a particular 4-vector or a special 4 by 4 matrix
- What do they do?
  - Addition and subtraction are just like vectors
  - Multiplication:

$$(a_1 + \underline{v}_1) \cdot (a_2 + \underline{v}_2) = a_1 \cdot a_2 - \left(\underline{v}_1^T \cdot \underline{v}_2\right) + a_1 \cdot \underline{v}_2 + a_2 \cdot \underline{v}_1 + \underline{v}_1 \times \underline{v}_2$$

  - Division: multiplication by the reciprocal of a quaternion

$$q = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$$

$$\frac{1}{a + \underline{v}} = \frac{1}{a^2 + |v|^2} \cdot (a - \underline{v})$$

Multiplication is Not Commutative

- Order of factors in multiplication is significant

$$\left(a_1+\underline{v}_1\right)\cdot\left(a_2+\underline{v}_2\right)=a_1\cdot a_2-\left(\underline{v}_1^T\cdot\underline{v}_2\right)+a_1\cdot\underline{v}_2+a_2\cdot\underline{v}_1+\underline{v}_1\times\underline{v}_2$$

$$\left(a_2+\underline{v}_2\right)\cdot\left(a_1+\underline{v}_1\right)=a_1\cdot a_2-\left(\underline{v}_1^T\cdot\underline{v}_2\right)+a_1\cdot\underline{v}_2+a_2\cdot\underline{v}_1-\underline{v}_1\times\underline{v}_2$$

- These products are NOT THE SAME
  - Unless

$$\underline{v}_1\times\underline{v}_2=\underline{0}$$

THIS IS IMPORTANT
We will get back to it later

- Sets of quaternions that all have the same vector axis
  - Coaxial quaternions form a field that is isometric to complex numbers
  - All complex arithmetic and analytic functions are valid in these fields

Where Did They Come From?

- First formulated as such in 1843 by William Rowan Hamilton in 1843
  - Inspiration carved on the side of Brougham Bridge in chalk:

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1$$

$$i \cdot j = k, \;\; j \cdot i = -k$$

$$j \cdot k = i, k \cdot j = -i$$

$$k \cdot i = j, \;\; i \cdot k = -j$$

Basic Concept is Vector Cross-Product

Why Are They Important?

- The algebra of rotating body coordinates
  - A method to characterize rotating coordinates of a point on a body
    - Nose and wing positions on an aircraft
    - Leading edge of a Frisbee
    - Direction Up/Down, positions of control fins of spinning missile
    - A point on the ground or another aircraft
  - From the standpoint of a ground observer or target
  - From the standpoint of the missile or aircraft
- Known principles are older
  - Euler's Rotation Theorem of 1775: Multiple rotations of a rigid body are equivalent to a rotation about a single axis
  - Coordinate rotation matrices use three rotations: roll, pitch, yaw

Other Uses of Quaternions

- Geometry
  - Plane and solid geometry, as an extension to vector algebra (see Hardy in the References)
  - Computer graphics, for their ability to rotate solid bodies
  - Computer vision, to provide
    - Rotation of a solid object
    - Movement of the solid object
    - Rotation and movement of the viewer point of view
- Crystallographic texture analysis (see References)
- Pure and applied mathematics
  - Cayley algebras (see References)

Quaternion Rotations are Used in

- Autopilots, to keep track of
  - The orientation of the platform
  - Angle of attack relative to aircraft motion
  - The field of view of the aircraft’s sensors
- Computer vision
  - To characterize the orientation of an object
  - To characterize the orientation of the viewer
- Tracking and estimation
  - To estimate the orientation of an object to model its flight dynamics
  - To estimate what a tracked object “sees”

The Key Capability is Characterizing Rotation

Some Vector Identities We Will Need

- Subspace operator, finds projection onto plane normal to $\underline{v}$

$$Sub_v = I - \frac{\underline{v}\cdot\underline{v}^T}{\left(\underline{v}^T\cdot\underline{v}\right)}$$

Identity Matrix

Matrix operator that extracts component of vector along $\underline{v}$

- Skew-symmetric form for use with cross-products

$$\underline{v}\times\underline{w} = S_v\cdot\underline{w}$$

$$S_v = \frac{\partial\left(\underline{v}\times\underline{w}\right)}{\partial\underline{w}} = \begin{bmatrix} 0 & -v_3 & +v_2 \\ +v_3 & 0 & -v_1 \\ -v_2 & +v_1 & 0 \end{bmatrix}, \quad S_v^2 = -\left(\underline{v}^T\cdot\underline{v}\right)\cdot Sub_v$$

Unit Vectors and Notation

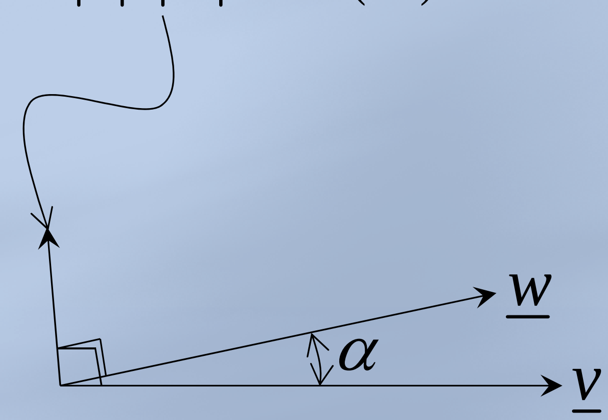
- Unit vector along a vector $\underline{v}$

$$\underline{u}_v = \frac{\underline{v}}{\left|\underline{v}\right|} = \frac{\underline{v}}{\sqrt{\left(\underline{v}^T \cdot \underline{v}\right)}}$$

- Right-hand-rule unit vector

$$\underline{v} \times \underline{w} = \left|\underline{v}\right| \cdot \left|\underline{w}\right| \cdot \sin\left(\alpha\right) \cdot \underline{u}_{Rvw}$$

$$\underline{v} \times \underline{w} = \left|\underline{v}\right| \cdot \left|\underline{w}\right| \cdot \sin\left(\alpha\right) \cdot \underline{u}_{Rvw}$$



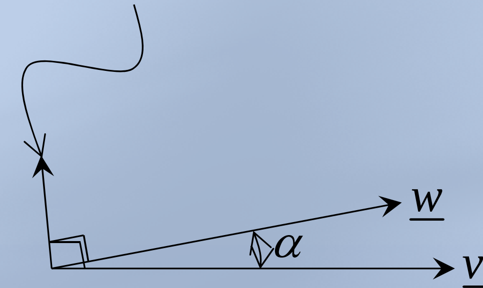
Two Ways of Interpreting $\underline{v} \times \underline{w} \times \underline{v}$

- The repeated cross product, geometric viewpoint

$$\underline{v}\times\left(\underline{v}\times\underline{w}\right) = \underline{v}\times\left(\left|\underline{v}\right|\cdot\left|\underline{w}\right|\cdot\sin\left(\alpha\right)\cdot\underline{u}_{Rvw}\right) = \left|\underline{v}\right|\cdot\left(\left|\underline{v}\right|\cdot\left|\underline{w}\right|\cdot\sin\left(\alpha\right)\right)\cdot\underline{u}_{Rv(vw)}$$

$$= -\left(\underline{v}^T\cdot\underline{v}\right)\cdot\left(I - \frac{\underline{v}\cdot\underline{v}^t}{\left(\underline{v}^T\cdot\underline{v}\right)}\right)\cdot\underline{w}$$

$$\underline{v}\times\underline{w} = \left|\underline{v}\right|\cdot\left|\underline{w}\right|\cdot\sin\left(\alpha\right)\cdot\underline{u}_{Rvw}$$



- Triple cross product, algebraic viewpoint

$$\left(\underline{v}\times\underline{w}\right)\times\underline{v} = \left(S_v\cdot\underline{w}\right)\times\underline{v} = -\underline{v}\times\left(S_v\cdot\underline{w}\right)$$

$$= -S_v\cdot\left(S_v\cdot\underline{w}\right) = -S_v^2\cdot\underline{w} = \left(\underline{v}^T\cdot\underline{v}\right)Sub_v\cdot\underline{w}$$

How Do Quaternions Rotate vectors?

- A few definitions of quaternion arithmetic
  - Consider a vector as a quaternion with a zero real part
  - Define the conjugate of a quaternion as reversing the sign of the vector part
- Left-multiply a vector by a quaternion
- Then right-multiply that result by 1/q

$$q \cdot \underline{v} = \left(a + \underline{b}\right) \cdot \underline{v}$$

$$= -\left(\underline{b}^T \cdot \underline{v}\right) + a \cdot \underline{v} + \underline{b} \times \underline{v}$$

$$q \cdot \underline{v} \cdot \frac{1}{q} = \frac{1}{a^2 + \left(\underline{b}^T \cdot \underline{b}\right)} \cdot \left(a^2 \cdot \underline{v} + \left(\underline{b}^T \cdot \underline{v}\right) \cdot \underline{b} + 2 \cdot a \cdot \left(\underline{b} \times \underline{v}\right) - \underline{b} \times \underline{v} \times \underline{b}\right)$$

A Huge Simplification (1 of 2)

- Use a quaternion defined using a rotation angle ϕ and axis *u*

$$q = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\cdot \underline{u}$$

- Then $q\cdot\underline{v}\cdot(1/q)$ becomes

$$q\cdot\underline{v}\cdot\frac{1}{q} = a^2\cdot\underline{v} + \left(\underline{b}^T\cdot\underline{v}\right)\cdot\underline{b} + 2\cdot a\cdot\left(\underline{b}\times\underline{v}\right) - \underline{b}\times\underline{v}\times\underline{b}$$

$$-\cos^2\left(\frac{\phi}{2}\right)\cdot\left(\underline{u}\cdot\underline{u}^T\right) + \cos^2\left(\frac{\phi}{2}\right)\cdot\left(\underline{u}\cdot\underline{u}^T\right)$$

$$= \cos^2\left(\frac{\phi}{2}\right)\cdot\underline{v} + \uparrow\sin^2\left(\frac{\phi}{2}\right)\cdot\left(\underline{u}\cdot\underline{u}^T\right)\cdot\underline{v} + \sin(\phi)\cdot\left(\underline{u}\times\underline{v}\right) - \sin^2\left(\frac{\phi}{2}\right)\cdot Sub_u\cdot\underline{v}$$

$$= \left(\underline{u}\cdot\underline{u}^T\right)\cdot\underline{v} + \cos(\phi)\cdot Sub_u\cdot\underline{v} + \sin\phi\cdot\left(\underline{u}\times\underline{v}\right)$$

A Huge Simplification (2 of 2)

- Interpreting this operation:

$$q \cdot \underline{v} \cdot \frac{1}{q} = \left(\underline{u} \cdot \underline{u}^T\right) \cdot \underline{v} + \cos\left(\phi\right) \cdot Sub_u \cdot \underline{v} + \sin\phi \cdot \left(\underline{u} \times \underline{v}\right)$$

- The first term $\left(\underline{u} \cdot \underline{u}^T\right) \cdot \underline{v}$
  - Extracts the component of <u>*v*</u> along <u>*u*</u>
  - This component of <u>*v*</u> is left unchanged
- The second term $\cos\left(\phi\right) \cdot Sub_u \cdot \underline{v}$
  - Finds the projection of <u>*v*</u> on plane normal to <u>*u*</u> times cos(ϕ)
- The third term $\sin\phi \cdot \left(\underline{u} \times \underline{v}\right)$
  - Finds the projection of <u>*v*</u> on plane normal to <u>*u*</u> times sin(ϕ)

Summary of Quaternion Rotation

- Given the rotation quaternion

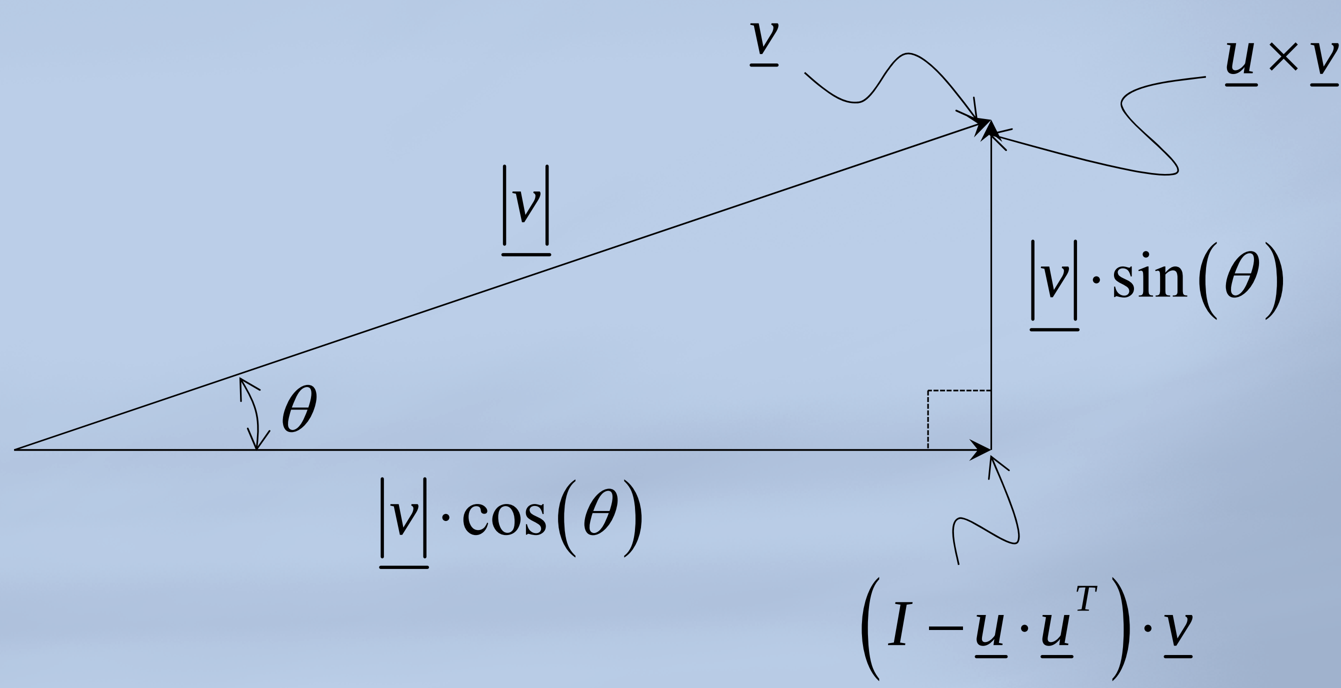
$$q = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) \cdot \underline{u}$$

$$q = \exp\left(\frac{\phi}{2} \cdot \underline{u}\right)$$

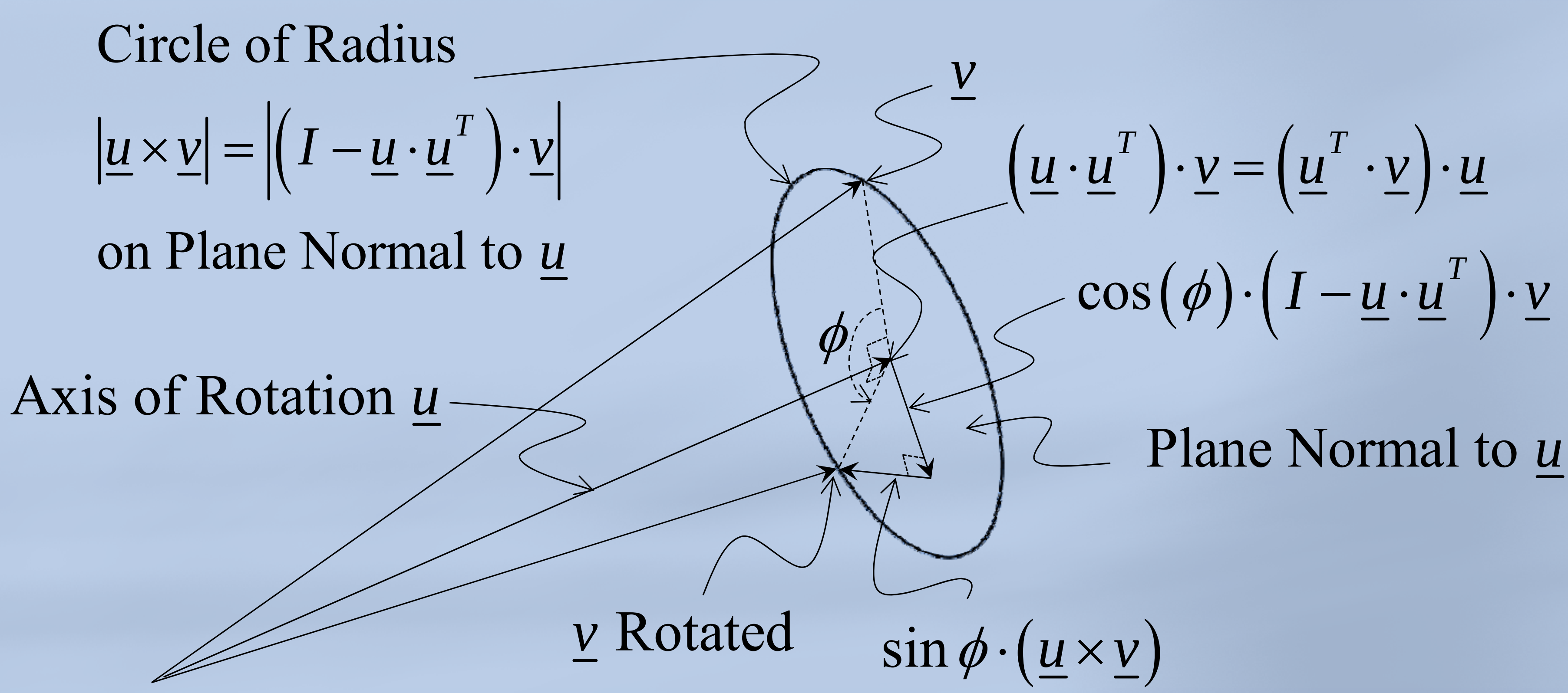
- Axis of rotation is $\underline{u}$
- Angle of rotation is ϕ
- Direction of rotation is by the right-hand-rule
- Range of the variable ϕ

$$-\pi < \phi \le \pi, \quad -\frac{\pi}{2} < \frac{\phi}{2} \le \frac{\pi}{2} \Leftrightarrow \cos\left(\frac{\phi}{2}\right) \ge 0$$

The Cone of Rotation



Geometry of Quaternion Rotation



Why are Quaternions Simpler?

- A rotation from the current right-handed Cartesian coordinate system is with a 3 X 3 matrix

$$A_{Rot} = \begin{bmatrix} \underline{u}_X^T \\ \underline{u}_Y^T \\ \underline{u}_Z^T \end{bmatrix}, \quad A_{Rot} \cdot \underline{v} = \begin{bmatrix} \left(\underline{u}_X^T \cdot \underline{v}\right) \\ \left(\underline{u}_Y^T \cdot \underline{v}\right) \\ \left(\underline{u}_Z^T \cdot \underline{v}\right) \end{bmatrix} = \underline{v}_{New}$$

- Need to use *A* when you have the quaternion?

$$A_{Rot}\left(q_0 + \underline{v}_q\right) = \frac{1}{|q|^2} \cdot \left[\left(q_0^2 - \left|\underline{v}_q\right|^2\right) \cdot I + 2 \cdot \left(q_0 \cdot S\left(\underline{v}_q\right) + \underline{v}_q \cdot \underline{v}_q^T\right)\right]$$

The Aerospace Sequence

- The Aerospace Sequence, rotating from ECEF to airframe coordinates
  - First rotate clockwise-looking-down about the Up axis to aircraft heading plus yaw,
  - Then rotate clockwise-bow-to-right about the aircraft pitch axis to the aircraft attitude,
  - Then rotate clockwise-looking-forward to aircraft roll angle.
- Order applied is Yaw, then Pitch, then Roll.
- To rotate from airframe to ECEF, order is reversed

Rotating To and From Other Coordinates

- To rotate from ECEF to airframe coordinates

$$\underline{v}_{Airframe} = q \cdot \underline{v}_{ECEF} \cdot q^*$$

- To rotate from airframe coordinates to ECEF

$$\underline{v}_{ECEF} = q^* \cdot \underline{v}_{Airframe} \cdot q$$

- Simplification: Quaternions don't require
  - Keeping track of the aerospace sequence
  - Maintenance of roll, pitch and yaw angles
  - Special provision for when pitch goes to or through ±π/2

Roll, Pitch and Yaw Quaternions

$$q_{Roll} = \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \\ 0 \\ 0 \end{bmatrix}, \quad q_{Pitch} = \begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) \\ 0 \\ \sin\left(\frac{\gamma}{2}\right) \\ 0 \end{bmatrix}, \quad q_{Yaw} = \begin{bmatrix} \cos\left(\frac{\psi}{2}\right) \\ 0 \\ 0 \\ \sin\left(\frac{\psi}{2}\right) \end{bmatrix}$$

$$q = q_{Yaw} \cdot q_{Pitch} \cdot q_{Roll}$$

Quaternion from Euler Angles

$$q = \begin{bmatrix} \cos\left(\frac{\phi}{2}\right)\cdot\cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\cdot\sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right)\cdot\cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) + \cos\left(\frac{\phi}{2}\right)\cdot\sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ \cos\left(\frac{\phi}{2}\right)\cdot\sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\cdot\cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right)\cdot\sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) + \cos\left(\frac{\phi}{2}\right)\cdot\cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \end{bmatrix}$$

Rotation Matrices from Roll, Pitch and Yaw

$$A_{Roll}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \sin(\phi) & \cos(\phi)\end{bmatrix}$$

$$A_{Pitch}=\begin{bmatrix}\cos(\gamma) & 0 & \sin(\gamma)\\ 0 & 1 & 0\\ -\sin(\gamma) & 0 & \cos(\gamma)\end{bmatrix}$$

$$A_{Yaw}=\begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A=A_{Roll}\cdot A_{Pitch}\cdot A_{Yaw}$$

Need Quaternion from Rotation Matrix? (1 of 2)

- We need these to find the quaternion from *A*

$$\mathrm{Sym}\{A\} = \frac{1}{2}\cdot\left(A + A^T\right),\ \mathrm{Asym}\{A\} = \frac{1}{2}\cdot\left(A - A^T\right)$$

$$\mathrm{trace}\{A\} = A_{1,1} + A_{2,2} + A_{3,3}$$

- The rotation quaternion axis $\underline{asv}_{Asym}$ is found from

$$\mathrm{Asym}\{A\} = \begin{bmatrix} 0 & -asv_3 & +asv_2 \\ +asv_3 & 0 & -asv_1 \\ -asv_2 & +asv_1 & 0 \end{bmatrix} \Leftrightarrow \underline{asv}_{Asym} = \begin{bmatrix} asv_1 \\ asv_2 \\ asv_3 \end{bmatrix} = \sin(\phi)\cdot Ss_u$$

Quaternion from Rotation Matrix (2 of 2)

- Sine and cosine of roll and axis vector from asymmetric Matrix

$$\underline{asv}_{Asym} = \begin{bmatrix} asv_1 \\ asv_2 \\ asv_3 \end{bmatrix} = \sin(\phi) \cdot Ss_u, \quad \cos(\phi) = \frac{trace\{A_{Sym}\} - 1}{2}$$

- Quaternion

$$\sin(\phi) = \left|\underline{asv}_{Asym}\right|, \; \underline{u} = \frac{\underline{asv}_{Asym}}{\left|\underline{asv}_{Asym}\right|}$$

$$z = \tan\left(\frac{\phi}{2}\right) = \frac{1 - \cos(\phi)}{\sin(\phi)} = \frac{\sin(\phi)}{1 + \cos(\phi)}$$

$$q = \frac{1 - z^2}{1 + z^2} + \frac{2 \cdot z}{1 + z^2} \cdot \underline{u}$$

Euler Angles from Quaternion

$$q_1 \cdot q_3 + q_2 \cdot q_4 = \frac{1}{2} \cdot \sin(\gamma)$$

Provides pitch

$$q_1 \cdot q_4 - q_2 \cdot q_3 = \frac{1}{2} \cdot \left(\cos(\gamma) \cdot \sin(\psi) \right)$$

$$q_1^2 + q_2^2 - q_3^2 - q_4^2 = \cos(\gamma) \cdot \cos(\psi)$$

Together with two-argument arctangent Provides yaw

Find roll, given pitch and yaw, from:

$$\begin{bmatrix} \cos\left(\frac{\gamma}{2}\right) \cdot \cos\left(\frac{\psi}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \cdot \sin\left(\frac{\psi}{2}\right) \\ \sin\left(\frac{\lambda}{2}\right) \cdot \sin\left(\frac{\psi}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \cdot \cos\left(\frac{\psi}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$$

From any two elements of q

Full Ambiguity Range of Roll from Quaternion

$$\begin{bmatrix} \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) & -\sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) & \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) \end{bmatrix}\cdot\begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$$

From any two elements of q

$$\begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} = \frac{2}{1+\cos(\gamma)\cdot\cos(\psi)}\cdot\begin{bmatrix} \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) & \sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ -\sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) & \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) \end{bmatrix}\cdot\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$$

$$\frac{\phi}{2} = \operatorname{atan2}\left(\begin{matrix} -\sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right)\cdot q_0 + \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right)\cdot q_1, \\ \cos\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right)\cdot q_0 + \sin\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right)\cdot q_1 \end{matrix} \right)$$

Given ϒ and ψ

Full Ambiguity Range of Roll from Quaternion

$$\begin{bmatrix} \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) & -\cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ \cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) & \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) \end{bmatrix}\cdot\begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} = \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

From any two elements of q

$$\begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \end{bmatrix} = \frac{2}{1-\cos(\gamma)\cdot\cos(\psi)}\cdot\begin{bmatrix} \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) & \cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) \\ -\cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right) & \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right) \end{bmatrix}\cdot\begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

$$\frac{\phi}{2} = \operatorname{atan2}\left(\begin{matrix} -\cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right)\cdot q_2 + \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right)\cdot q_2, \\ \sin\left(\frac{\gamma}{2}\right)\cdot\cos\left(\frac{\psi}{2}\right)\cdot q_2 + \cos\left(\frac{\gamma}{2}\right)\cdot\sin\left(\frac{\psi}{2}\right)\cdot q_3 \end{matrix} \right)$$

Given ϒ and ψ

Rotation Matrix and The Euler Angles

- Rotation matrix in terms of Euler angles
  - Product of roll, pitch, yaw rotation matrices

$$A = \begin{bmatrix} \cos(\gamma)\cdot\cos(\psi) & -\cos(\gamma)\cdot\sin(\psi) & \sin(\gamma) \\ \sin(\phi)\cdot\sin(\gamma)\cdot\cos(\psi)+\cos(\phi)\cdot\sin(\psi) & -\sin(\phi)\cdot\sin(\gamma)\cdot\sin(\psi)+\cos(\phi)\cdot\cos(\psi) & -\sin(\phi)\cdot\cos(\gamma) \\ -\cos(\phi)\cdot\sin(\gamma)\cdot\cos(\psi)+\sin(\phi)\cdot\sin(\psi) & \cos(\phi)\cdot\sin(\gamma)\cdot\sin(\psi)+\sin(\phi)\cdot\cos(\psi) & \cos(\phi)\cdot\cos(\gamma) \end{bmatrix}$$

- Euler angles from rotation matrix

$$\psi = \operatorname{atan2}\left(A_{12}, A_{11}\right)$$

$$\gamma = \cos^{-1}\left(\sqrt{A_{11}^2 + A_{12}^2}\right) = \sin^{-1}\left(A_{13}\right) = \operatorname{atan2}\left(A_{13}, \sqrt{A_{11}^2 + A_{12}^2}\right)$$

$$\phi = \operatorname{atan2}\left(-A_{23}, A_{33}\right)$$

The Orbital Element Sequence

- Reference frame is ECIC
  - X axis through vernal equinox (in Aries)
  - Z axis through North pole
  - Y axis is cross-product of Z axis with Y axis to give a right-handed system
- Translation to orbital elements coordinate system
  - First, rotation in longitude, positive East to the line of nodes (the longitude of the ascending node, or the point above which the satellite passes through the equatorial plane Northbound)
  - Then, inclination of the orbital plane, positive Eastward half plane upward
  - Then, true anomaly or angle from that point to the new X axis positive Northward.

Common Platform Coordinate Systems

- The Aerospace Sequence
  - Called the zyx sequence
  - Rotating base coordinates in order of yaw, pitch, then roll
  - Usually used for airborne objects from ECEF
- The Orbital Element Sequence
  - Called the zxz sequence
    - Rotating in longitude to the line of nodes
    - Then inclination of the orbital plane
    - Rotation to true anomaly
  - Usually used for LEO and MEO orbital object positions from ECIC
- Others (see Minkler and Minkler in References)

What About Equations of Rotational Motion?

- We begin with the moment of inertia matrix
- The next step is the angular momentum vector
- Outside forces are torque on the body
- Generality requires a differential equation
- A differentiation provides the rate of change of the angular momentum
- The resulting differential equation are
  - The equations of rotational motion
  - Classically called Euler's equations

The Moment of Inertia Matrix

- Moment of inertia matrix

$$M = \int \left(\left(\underline{x}^T \cdot \underline{x} \right) \cdot I - \underline{x} \cdot \underline{x}^T \right) \cdot \rho\left(\underline{x}\right) \cdot d\,\underline{x}$$

$$\underline{x} \cdot \underline{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$\left(\underline{x}^T \cdot \underline{x} \right) \cdot I - \underline{x} \cdot \underline{x}^T = \begin{bmatrix} x_2^2 + x_3^2 & -x_1 \cdot x_2 & -x_1 \cdot x_3 \\ -x_2 \cdot x_1 & x_1^2 + x_3^2 & -x_2 \cdot x_3 \\ -x_3 \cdot x_1 & -x_3 \cdot x_2 & x_1^3 + x_2^2 \end{bmatrix}$$

- Origin of coordinate system
  - Center of gravity of the body
  - HOLD THAT THOUGHT
- Result is a real, symmetrical positive definite matrix
- Singular value decomposition provides
  - Eigenvectors are axes of rotation
  - Eigenvalues form a diagonal moment of inertia matrix

WE ARE IN BODY COORDINATES

Angular Momentum

- In any coordinate system *r*, the analog of *m·v* is

$$\underline{h}_r = M \cdot \underline{\omega}_r$$

- The time derivative of angular momentum is

$$\frac{d}{dt}\underline{h} = M \cdot \left(\frac{d}{dt}\underline{\omega}_r \right) + \underline{to}$$

- Torque is the sum of lever arms crossed into force vectors

$$\underline{to} = \int \underline{r} \times d\,\underline{f}\left(\underline{r}\right) = \int S_r \cdot d\,\underline{f}$$

- Lever arm $\underline{r}$ is vector from axis to point where force is applied

Time Derivative of the Rotation Quaternion

- Finding equation for time derivative of quaternion

$$q^* \cdot q = 1$$

$$\frac{d}{dt} q^* \cdot q + q^* \cdot \frac{d}{dt} q = 0, \quad \frac{d}{dt} q^* \cdot q = -q^* \cdot \frac{d}{dt} q$$

- Conjugating a quaternion to produce the negative of the same quaternion means that we have a pure vector
- Derivative of a vector rotated from the body coordinates to the reference coordinate system

$$\frac{d}{dt} \underline{r}_r = \frac{d}{dt}\left(q \cdot \underline{r}_b \cdot q^*\right) = \frac{d}{dt} q \cdot \underline{r}_b \cdot q^* + q \cdot \underline{r}_b \cdot \frac{d}{dt} q^*$$

Getting to a Cross-Product

- Rotating the velocity in the reference coordinate system back to the body coordinates

$$\underline{v}_r = q^* \cdot \frac{d}{dt}\underline{r}_r \cdot q = q^* \cdot \frac{d}{dt} q \cdot \underline{r}_b + \underline{r}_b \cdot \frac{d}{dt} q^* \cdot q$$

- Fundamental identity from multiplication of quaternions

$$\frac{1}{2} \cdot \left(\underline{v}_1 \cdot \underline{v}_2 - \underline{v}_2 \cdot \underline{v}_1 \right) = \underline{v}_1 \times \underline{v}_2$$

- So that

$$\underline{v}_r = 2 \cdot \left(q^* \cdot \frac{d}{dt} q \right) \times \underline{r}_b = \underline{\omega}_b \times \underline{r}_b, \quad \underline{\omega}_b = 2 \cdot \left(q^* \cdot \frac{d}{dt} q \right)$$

Euler's Equations

- Rotating the angular momentum to the reference frame

$$q \cdot \underline{h}_r \cdot q^* = q \cdot M \cdot \underline{\omega}_b \cdot q^*$$

- Taking the derivative with respect to time

$$\frac{d}{dt} q \cdot \left[M \cdot \underline{\omega}_b \right] \cdot q^* + q^* \cdot \left[M \cdot \underline{\omega}_b \right] \cdot \frac{d}{dt} q^* + q \cdot \left[M \cdot \frac{d}{dt} \underline{\omega}_b \right] \cdot q^*$$

- Rotating back and solving for the time derivative of the rotation vector

$$M \cdot \frac{d}{dt} \underline{\omega}_b = \underline{to}_b - q^* \cdot \frac{d}{dt} q \cdot \left[M \cdot \underline{\omega}_b \right] - \left[M \cdot \underline{\omega}_b \right] \cdot \frac{d}{dt} q^* \cdot q$$

$$= \underline{to}_b - S_{\omega b} \cdot M \cdot \underline{\omega}_b$$

Equation for Numerical Solutions

- Euler's Equation for Motion of a Rotating Rigid Body

$$\frac{d}{dt}\underline{\omega}_b = M^{-1}\cdot\left(\underline{to}_b - S_{\omega b}\cdot M\cdot\underline{\omega}_b\right)$$

$$\frac{d}{dt}q = \frac{1}{2}\cdot q^{*}\cdot\underline{\omega}_b$$

- Time differential equation for the rotation matrix

$$\frac{d}{dt}A = A\cdot S_{\omega b}$$

Why are Quaternions More Accurate

- Sensitivity of rotation matrix are similar
  - WRT Roll, Pitch, Yaw
  - WRT q, vq
- Euler's equations & sensitivities
  - Build into your equations an exponential trend toward normalization
  - For quaternions, this is

Adjust for application

$$qstab(q) = qstabconst \cdot (|q| - 1), \quad \underline{\omega a} = \begin{bmatrix} qstab(q) \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad qstabconst = 1.0$$

Reasons for Including the Stability Term

- Robustness of simulation for unlimited run times
- Eliminates a software maintenance area
- Write-and-forget enabler
  - For critical systems functional blocks
  - For embedded software
  - Trouble-free components of larger models
- Quality attribute for delivered software
  - You won't hear from "quaternion magnitude decay"
  - Confidence by others in using your models
  - Robustness when others use it for non-predicted applications

Sources of Error with Quaternions

- Numerical errors in Euler's equations

$$\frac{d}{dt}\underline{\omega}_b = M^{-1}\cdot\left(\underline{to}_b - S_{\omega b}\cdot M\cdot\underline{\omega}_b\right)$$

$$\frac{d}{dt}q = \frac{1}{2}\cdot q^{*}\cdot\underline{\omega}_b$$

Add a stability term to the angular velocity term

- Numerical errors in rotation

$$\underline{v} = q\cdot\underline{v}_b\cdot\frac{1}{q} = \frac{1}{a^2+\left(\underline{b}^T\cdot\underline{b}\right)}\cdot\left(a^2\cdot\underline{v}_b+\left(\underline{b}^T\cdot\underline{v}_b\right)\cdot\underline{b}+2\cdot a\cdot\left(\underline{b}\times\underline{v}_b\right)-\underline{b}\times\underline{v}_b\times\underline{b}\right)$$

- All these equations are well-conditioned numerically

Sources of Errors with Direction Cosines

- Numerical Error in Euler's Equations

$$\frac{d}{dt}\underline{\omega}_b = M^{-1}\cdot\left(\underline{to}_b - S_{\omega b}\cdot M\cdot\underline{\omega}_b\right)$$

$$\frac{d}{dt}A = A\cdot S_{\omega b}$$

Keeping *A* unitary is complicated

- Numerical Error in Rotations

Differential equations in Euler angles are complicated

$$\underline{v} = A\cdot\underline{v}_b$$

- Everything is noisier when |ϒ| is near π/2
- "Gimbal lock" singularity at |ϒ| = π/2

Problems in Common with Both Approaches

- Interfaces
  - Different system blocks have a documented communication interface
  - The quantities in the interface are specified by the Interface Control Document (ICD), a systems engineering artifact
    - All quantities passed between system blocks are defined
    - Word length, normalization, physical units, data rate, static reference values such as the gravitational constant are in the ICD but not necessarily on the bus
    - This may include Euler angles or quaternion, or both
    - Aerospace sequence, orbital element sequence, etc. must be defined in ICD
- Coordinates must be exchanged and updated
  - Different system functions use different coordinate systems
  - Underlying coordinates for most systems must be inertial

Lets Look at Position: Coordinates for a Radar

- Base system coordinate system is ECIC
- Local coordinate system is radar coordinates
  - Origin is at the antenna phase center
  - X is horizontal, to left looking out from radar
  - Y is vertical, parallel to antenna face
  - Z is normal to plane of antenna, out radar axis
  - Very natural for planar radar antenna arrays
  - Not an inertial coordinate system
- $\underline{u}$ is line-of-sight from radar to target
- Position is what is characterized by the quaternion

F-35 AESA photo By Daderot - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=34902920

The Variables

- Position

$$q_{TC} = \ln\left(\frac{q_{\text{Target}}}{R_0}\right) = \ln\left(\frac{R_{\text{Target}}}{R_0}\right) + \frac{\pi}{2}\cdot \underline{u}_{\text{Target}}$$

$$\underline{u}_{\text{Target}} = \begin{bmatrix} u_{Left} \\ u_{Up} \\ \sqrt{1-u_{Left}^2-u_{Up}^2} \end{bmatrix}$$

We are using quaternions that share $\underline{u}$ as a field analogous to complex numbers

- Radar measures R_{Target}, u_{Left}, u_{Up}, $\dot{R}_{Target}$

The State Vector $\underline{x}$ and the Measurements $\underline{y}$

$$\frac{\partial r}{\partial \ln(r)} = \frac{1}{\left(\frac{\partial \ln(r)}{\partial r}\right)} = \frac{1}{\left(\frac{1}{r}\right)} = r = \frac{R}{R_0}$$

$$\underline{x} = \begin{bmatrix} \ln\left(\frac{R}{R_0}\right) \\ u_{Left} \\ u_{Up} \\ \frac{\dot{R}}{R} \\ \dot{u}_{Left} \\ \dot{u}_{Up} \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} \frac{R}{R_0} \\ u_{Left} \\ u_{Up} \\ \frac{\dot{R}}{R} \end{bmatrix}$$

$$H = \frac{\partial \underline{y}}{\partial \underline{x}} = \begin{bmatrix} \frac{R}{R_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

H is the Sensitivity Matrix

State Transition Matrix

- General form

$$\Phi(\tau,t)=\frac{\partial \underline{x}(t+\tau)}{\partial \underline{x}(t)}\approx\begin{bmatrix}1&0&0&\tau&0&0\\0&1&0&0&\tau&0\\0&0&1&0&0&\tau\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

- Actual exact form depends on target motion model

Consequences of Selection of States

- Statistical efficiency
  - Linear relationship between measurements and position states because measurements and position states, range rate state are the same as the measurements
- Numerical efficiency
  - Kalman filter base equations all have matrices with unitless elements of same general magnitude
  - Advantages accrue to
    - Joseph Stabilized Form
    - UDUT Square Root Filter
    - SRIF

$$\underline{\tilde{x}} = \Phi \cdot \underline{\hat{x}}_-$$

$$\tilde{P} = \Phi \cdot P_- \cdot \Phi^T + Q, \quad P^{-1} = \tilde{P}^{-1} + H^T \cdot R^{-1} \cdot H$$

$$K = P \cdot H^T \cdot R^{-1}$$

$$\underline{\hat{x}} = \underline{\tilde{x}} + K \cdot \left(\underline{y} - \underline{h}\left(\underline{\tilde{x}} \right) \right)$$

Kalman Filter Types

- Joseph stabilized form (Gelb, pp 305-306)

$$\tilde{P} = \Phi \cdot P_{-} \cdot \Phi^{T} + Q, \quad K = \tilde{P} \cdot H^{T} \cdot \left(H \cdot \tilde{P} \cdot H^{T} + R \right)^{-1}$$

$$P = \left(I - K \cdot H \right) \cdot \tilde{P} \cdot \left(I - K \cdot H \right)^{T} + K \cdot R \cdot K^{T}$$

- UDUT Factorization
  - Uses "square root" of covariance matrix

$$U \cdot D \cdot U^{T} = P, \quad U \text{ upper triangular w/1s on diagonal, } D \text{ diagonal}$$

  - Nearly a drop-in upgrade for Joseph stabilized form
- Square root information filter (SRIF)

Square Root Information Filter

- Works with a Cholesky factorization of inverse of covariance matrix
- Most number crunching is done using Householder reflections
  - Left-multiplication by Householder reflections, matrices of the form

$$T = I - 2 \cdot \underline{u} \cdot \underline{u}^T$$

  - Well known for excellent numerical properties
- Accuracy and numerical advantages when
  - Best model at start is initialization with “infinite variance” of unobservable states
  - One or more states is poorly observed for several update periods at the beginning of track
  - Anytime one or more states are carried along without observability
  - Huge numbers of data points are used in updates (rare in radar trackers)

The Subtleties of Tracking Suborbital Objects

- Target position updates
  - Atmospheric object target motion model and updates in ECEF
  - Exoatmospheric object target motion model and updates updates in ECIC
  - Some use custom updates
    - Fast-rising missiles exhibit Coriolis from ECEF rotation
    - High exoatmospheric objects need custom dynamic modeling
- ECEF rotates with time and must be periodically updated
  - Long wait times without updates in ECEF result in gravity "down" rotating 15 degrees an hour
  - This resulted in Patriot missiles missing a Scud in first Iraq war

Examples

- Mathcad simulation of ICBM payload re-entry cone
  - Mathcad Program (start)
  - Empty cone (start)
  - Empty with radar fuze window (start)
  - With warhead mass and radar fuze window (start)

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