Bayliss Objective Function

- Objective Function
  - Begin with Airy Disk for asymmetrical pattern

$$b_{BObj}(su) = \frac{\pi \cdot su}{S} \cdot \operatorname{sinc}\left(\sqrt{\pi \cdot su - A^2}\right)$$

  - Move first few zeros near split main lobe to define equiripple sidelobes
- Adjust S, A to achieve specified sidelobe level from peak of split main lobe
- Result is numerical, not a simple elementary function

Form of Objective Function

- Use the Product-of-Zeros Format

$$bs_{BObj}(su) = \frac{\pi \cdot su}{S} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{\left((\pi \cdot su)^2 - A^2 \right)}{(\pi \cdot k)^2} \right)$$

- After zeros moved

$$b_{BObj}(su) = C \cdot su \cdot \prod_{k=1}^{\infty} \left(1 - \frac{(\pi \cdot su)^2}{\left(\pi \cdot z_k^2\right)} \right)$$

- All parameters A, S, C, z_k are heuristic

The Problem

- Define the scaling as a function of n bar, the zero you hold in common to match up the functions:

$$\sigma = \frac{\mu_{\bar{n}}}{z\left(\bar{n}\right)} \underset{\bar{n}\to\infty}{\approx} \frac{\bar{n} - 1/4}{\bar{n}} < 1$$

- The scaling is less than 1 for *all* n bar, increases asymptotically to 1
  - Not apparent from looking at zeros of objective function before heuristic tweaking of sidelobes
  - Not recognized by Bayliss in his original BSTJ paper

Bayliss' BSTJ Paper

Possible Solutions

- Take What You Can Get and Tweak
- Change the Asymptotic Function
- Take the form of the final solution

$$wb(rs)\begin{cases} = Cb\cdot\cos(\phi)\displaystyle\sum_{k=1}^{\bar{n}-1} bt_k \cdot J_1(\pi\cdot\mu_k\cdot rs), \; rs<1 \\ = 0, \; rs>1 \end{cases}$$

  - Consider the vector of coefficients <u>bt</u> as a variable in an optimization process, experiment with n bar

New Base Objective Function

- Abramowitz & Stegun p. 370 no. 9.5.11

$$J_{\nu}'(\pi \cdot z) = \frac{\left(\frac{\pi \cdot z}{2}\right)^{\nu-1}}{2 \cdot \Gamma(\nu)} \cdot \prod_{s=1}^{\infty}\left(1 - \frac{z^2}{\mu_{s,\nu}^2}\right)$$

- Our Function

$$su \cdot J_1'(\pi \cdot su) = \frac{su}{2} \cdot \prod_{s=1}^{\infty}\left(1 - \frac{su^2}{\mu_s^2}\right)$$

- Bayliss' Eq. (21) implies nbump=1

New Asymptotic Function

- Eliminate the first zero, or two, or three…

$$b_{BAsyp}\left(\underline{su}\right)=\frac{su\cdot J_1'\left(\pi\cdot su\right)}{2\cdot\prod_{n=1}^{nbump}\left(1-\frac{\left(\pi\cdot su\right)^2}{\mu_n^2}\right)}$$

- This allows matching up zero number n bar with the second, or even third or fourth zero of $J_1'(z)$ to get $\sigma>1$

Bayliss Theoretical Beam Pattern

- Product of Zeros Form

$$bb(su) = 2\pi \cdot su \cdot J_1'(\pi \cdot su) \cdot \frac{\prod_{n=1}^{\bar{n}-1}\left(1 - \frac{su^2}{\sigma^2 \cdot z_n^2}\right)}{\prod_{n=1}^{\bar{n}+nbump-1}\left(1 - \frac{su^2}{\mu_n^2}\right)}$$

- Note that the first nbump zeros are simply canceled, the next n bar zeros are moved

Bessel Function Interpolation

$$f(z)=\sum_{k=1}^{\bar{n}+nbump-1}\frac{1}{J1N_k}\cdot\frac{z}{\mu_k}\cdot\frac{J_1'(\pi\cdot z)}{\left(1-\dfrac{z^2}{\mu_k^2}\right)}\cdot f(\pi\cdot\mu_k)$$

$$\frac{d}{dz}J_1'(\pi\cdot\mu_k)=-\pi\cdot\left(1-\frac{1}{\pi^2\cdot\mu_k^2}\right)\cdot J_1(\pi\cdot\mu_k)$$

$$J1N_k=\lim_{z\to\mu_k}\left(\frac{J_1'(\pi\cdot z)}{\left(1-\left(\dfrac{z}{\mu_k}\right)^2\right)}\right)=\frac{\pi\cdot\mu_k}{2}\left(1-\frac{1}{\pi^2\cdot\mu_k^2}\right)\cdot J_1(\pi\cdot\mu_k)$$

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The Bayliss Ideal Beam Pattern

$$f\left(su\right)=\sum_{k=1}^{\bar{n}+nbump-1} 2\cdot\pi\cdot\mu_k\cdot\frac{\prod_{n=1}^{\bar{n}-1}\left(1-\frac{\mu_k^2}{\sigma^2\cdot z_n^2}\right)}{\prod_{\substack{n=1\\ n\neq k}}^{\bar{n}+nbump-1}\left(1-\frac{\mu_k^2}{\mu_n^2}\right)}\cdot\frac{su}{\mu_k}\cdot\frac{J_1'\left(\pi\cdot su\right)}{\left(1-\frac{z^2}{\mu_k^2}\right)}$$

Inverting the Bessel Transform

Abramowitz & Stegun, p. 484, Eq. 11.3.28

$$\int^{z}\left\{\left(k^2-l^2\right)\cdot t-\frac{\mu^2-\nu^2}{t}\right\}\cdot C_{\mu}\left(k\cdot t\right)\cdot D_{\nu}\left(l\cdot t\right)\cdot dt$$

$$=z\cdot\left\{k\cdot C_{\mu+1}\left(k\cdot z\right)\cdot D_{\nu}\left(l\cdot z\right)-l\cdot C_{\mu}\left(k\cdot z\right)\cdot D_{\nu+1}\left(l\cdot z\right)\right\}$$

$$-\left(\mu-\nu\right)\cdot C_{\mu}\left(k\cdot z\right)\cdot D_{\nu}\left(l\cdot z\right)$$

Our variables (used with order 0 for the Taylor window, order 1 in the next slide)

$$k\to\pi\cdot\mu_k,\quad l\to\pi\cdot su,\quad t\to r,\quad z=1,\quad C_1\left(\pi\cdot\mu_k\right)=D_1\left(\pi\cdot\mu_k\right)=J_1\left(\pi\cdot\mu_k\right)$$

Our integral

$$\int_0^1 J_1\left(\pi\cdot\mu_k\cdot r\right)\cdot r\cdot J_1\left(\pi\cdot su\cdot r\right)\cdot dr$$

$$=\frac{\pi\cdot\mu_k\cdot J_2\left(\pi\cdot\mu_k\right)\cdot J_1\left(\pi\cdot su\right)\cdot\mu_k-\pi\cdot su\cdot J_1\left(\pi\cdot\mu_k\right)\cdot J_2\left(\pi\cdot su\right)}{\pi^2\cdot\mu_k^2-\pi^2\cdot su^2}$$

Bessel Transform Pair

$$\int_0^1 J_1\left(\pi \cdot \mu_k \cdot r\right) \cdot r \cdot J_1\left(\pi \cdot su \cdot r\right) \cdot dr = \frac{J_1\left(\pi \cdot \mu_k\right)}{\pi \cdot \mu_k} \cdot \frac{su}{\mu_k} \cdot \frac{J_1'\left(\pi \cdot su\right)}{1 - \frac{su^2}{\mu_k^2}}$$

$$\int_0^\infty \frac{su}{\mu_k} \cdot \frac{J_1'\left(\pi \cdot su\right)}{1 - \frac{su^2}{\mu_k^2}} \cdot su \cdot J_1\left(\pi \cdot su \cdot r\right) \cdot dsu = \frac{\pi \cdot \mu_k}{J_1\left(\pi \cdot \mu_k\right)} \cdot J_1\left(\pi \cdot \mu_k \cdot r\right)$$

Note that $J_1'\left(z\right) = J_0\left(z\right) - \frac{J_1\left(z\right)}{z}$

Bayliss Weighting Function

$$wb(r) = \sum_{k=1}^{\bar{n}+nbump-1} b_k \cdot J_1(\pi \cdot \mu_k \cdot r)$$

$$b_k = \frac{\pi \cdot \mu_k^2}{J_1(\pi \cdot \mu_k)} \cdot \frac{\prod_{n=1}^{\bar{n}-1} \left(1 - \frac{\mu_k^2}{\sigma^2 \cdot z_n^2}\right)}{\prod_{\substack{n=1 \\ n \neq k}}^{\bar{n}+nbump-1} \left(1 - \frac{\mu_k^2}{\mu_n^2}\right)}$$