The Mathematics of the Taylor Window

- The origins of Taylor weighting
  - Beam pattern of unweighted circular aperture; the Airy disk
  - Beam pattern of Chebychev, limiting case for infinite element density
  - Original 2-D Chebychev innovation by Y. V. Baklanov, 1966.
- Variables:

$$S = 10^{Sidelobes(dB)/20}$$

$$A = \cosh^{-1}(S)$$

$$\underline{su} = \frac{k \cdot D \cdot \underline{u}}{2\pi} = \frac{D}{\lambda} \cdot \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\underline{rs} = \begin{bmatrix} \frac{rx}{D_x} \\ \frac{ry}{D_y} \end{bmatrix} = \begin{bmatrix} ud_x \\ ud_y \end{bmatrix}$$

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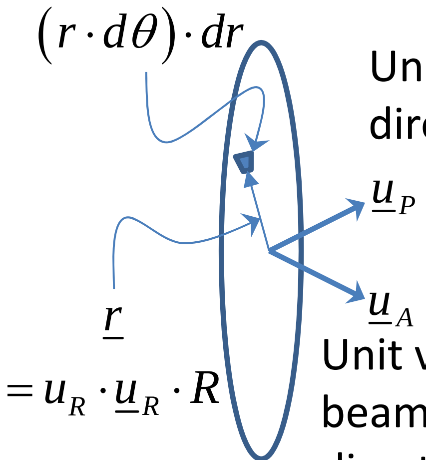
Pattern of Weighted Circular Aperture



Unit vector in direction of pattern

Unit vector in beam pointing direction

Field strength in direction of pattern:

$$b\left(u\right) = \frac{1}{\pi \cdot R^2} \cdot \int_{r=0}^{R} \int_{\theta=-\pi}^{\pi} W\left(r,\theta\right) \cdot \exp\left(-j \cdot \frac{2\pi}{\lambda} \cdot r \cdot u \cdot \cos\left(\theta\right)\right) \cdot r \cdot dr \cdot d\theta$$

$$= \frac{2}{R^2} \cdot \int_{0}^{R} W\left(r\right) \cdot r \cdot J_0\left(k \cdot r \cdot u\right) \cdot dr$$

The weighting from the beam pattern:

$$W\left(r\right) = k^2 \cdot \int_{0}^{\infty} b\left(u\right) \cdot u \cdot J_0\left(k \cdot r \cdot u\right) \cdot du$$

Identities

$$u \cdot \cos\left(\theta\right) = \underline{u}_R^T \cdot \left(\underline{u}_P - \underline{u}_A\right) \qquad k = \frac{2\pi}{\lambda}$$

$$\int_{-\pi}^{\pi} \cos\left(n \cdot \theta\right) \cdot \exp\left(-j \cdot k \cdot r \cdot u \cdot \cos\left(\theta\right)\right) \cdot d\theta = \left(j\right)^n \cdot 2\pi \cdot J_n\left(k \cdot r \cdot u\right)$$

Objective Function: Circular Aperture

- The Airy disk in direction cosines

$$b_{Airy}(u) = 2 \cdot \frac{J_1(k \cdot R \cdot u)}{k \cdot R \cdot u}, W(r) \begin{cases} = 1, r < R \\ = 0, r > R \end{cases}$$

- In terms of the scaled direction cosine

$$k \cdot R \cdot u = \frac{2\pi}{\lambda} \cdot R \cdot u = \pi \cdot \frac{2 \cdot R}{\lambda} \cdot u = \pi \cdot |su|$$

$$b_{Airy}\left(\left|\underline{su}\right|\right) = 2 \cdot \frac{J_1\left(\pi \cdot \left|\underline{su}\right|\right)}{\pi \cdot \left|\underline{su}\right|}$$

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Taylor's Objective Function

- An Unrealizable equiripple pattern

$$b\left(\underline{su}\right)=\begin{cases}\dfrac{1}{S}\cdot\cosh\left(\sqrt{A^2-\left(\pi\cdot\left|\underline{su}\right|\right)^2}\right), \left|\underline{su}\right|^2<=\dfrac{A}{\pi}\\ \dfrac{1}{S}\cdot\cos\left(\sqrt{\left(\pi\cdot\left|\underline{su}\right|\right)^2-A^2}\right), \left|\underline{su}\right|^2>=\dfrac{A}{\pi}\end{cases}$$

- NOTE: Beam weight is zero outside of circular aperture, has infinite circular "spike" on perimeter

Mathematical Identities

- Cosine and Bessel function $J_1(\pi z)$ as an infinite product defining their zeros:

$$J_1(\pi \cdot z) = \frac{\pi \cdot z}{2} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\mu_k^2} \right), \ J_1(\pi \cdot \mu_k) = 0$$

$$\cos(\pi \cdot z) = \prod_{k=1}^{\infty} \left(1 - \frac{4 \cdot z^2}{(2 \cdot k - 1)^2} \right)$$

The Idea

- The Chebychev weighting produces the narrowest possible beam for a given aperture
- The Airy disk provides a realizable rolloff of sidelobes
- Taylor's innovation: Scale the objective function wider so that its zero number "n bar" coincides with the same zero in the Airy disk pattern

The Problems

- Define the scaling, squared, as a function of n bar, the zero you hold in common to match up the functions:

$$\sigma^2 = \frac{\mu_{\bar{n}}^2}{\left(\frac{A}{\pi}\right)^2 + \left(\bar{n} - \frac{1}{2}\right)^2}$$

- The scaling is less than 1 for small n bar, increases to a maximum, then decreases asymptotically to 1

Setting N Bar and the Scaling

- If the scaling is not greater than 1, a function that performs according to design is not possible
- If n bar is too large, the weighting function will develop spikes around the perimeter
- Conclusion: Good performance of the resulting weighting is found by picking n bar at or just over the peak scaling value.

Finding N Bar

- For large k, the zeros of the Bessel function approach a limit:

$$\mu_k \sim n + \frac{1}{4}$$

- This allows simple calculus to give us a rule of thumb (adding two gets good results):

$$\bar{n} \approx \frac{4}{3} \cdot \left(\frac{A}{\pi} \right)^2 + \frac{1}{2} \approx 1 + \left(\frac{Sidelobes(dB)}{22.4} \right)^2$$

- Best solution: do a simple search!

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The Taylor Ideal Beam Pattern

- We use the Airy disk beam pattern with the first n bar minus 1 zeros replaced by those of the scaled Chebychev beam pattern

$$bt\left(\underline{su}\right)=\frac{2\cdot J_1\left(\pi\cdot\left|\underline{su}\right|\right)}{\pi\cdot\left|\underline{su}\right|}\cdot\prod_{n=1}^{\bar{n}-1}\frac{\left(1-\frac{\left|su\right|^2}{\sigma^2\cdot\left(\left(\frac{A}{\pi}\right)^2+\left(n-\frac{1}{2}\right)^2\right)}\right)}{\left(1-\frac{\left|su\right|^2}{\mu_n^2}\right)}$$

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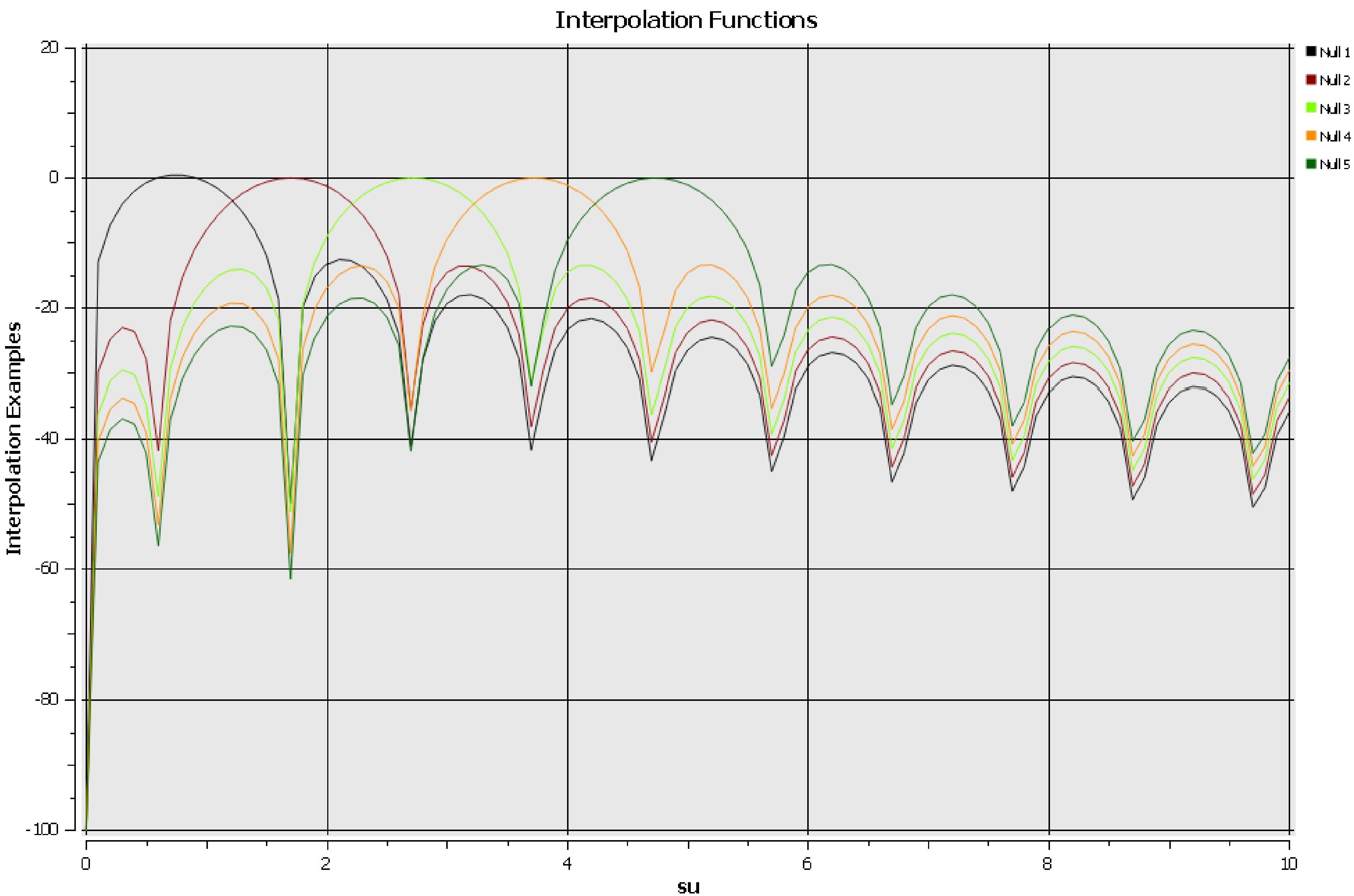
Bessel Function Interpolation

$$f(z) = \frac{2 \cdot J_1(\pi \cdot z)}{\pi \cdot z} \cdot f(0) + \sum_{k=1}^{\bar{n}-1} \frac{1}{J1N_k} \cdot \frac{z}{\mu_k} \cdot \frac{J_1(\pi \cdot z)}{\left(1 - \frac{z^2}{\mu_k^2}\right)} \cdot f(\mu_k)$$

$$J1N_k = \lim_{z \to \mu_k} \left(\frac{J_1(\pi \cdot z)}{\left(1 - \left(\frac{z}{\mu_k}\right)^2\right)} \right) = -\frac{\pi \cdot \mu_k \cdot J_0(\pi \cdot \mu_k)}{2}$$


Interpolation Functions
20
0
-20
-40
-60
-80
-100
Interpolation Examples
0
2
4
6
8
10
su
Null 1
Null 2
Null 3
Null 4
Null 5

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Taylor Beam Pattern as a Sum

$$bt\left(\underline{su}\right)=\frac{2\cdot J_1\left(\pi\cdot\left|\underline{su}\right|\right)}{\pi\cdot\left|\underline{su}\right|}+\sum_{k=1}^{\bar{n}-1}b_k\cdot\frac{\left|\underline{su}\right|}{\mu_k}\cdot\frac{J_1\left(\pi\cdot\left|\underline{su}\right|\right)}{\left(1-\frac{\left|\underline{su}\right|^2}{\mu_k^2}\right)}$$

$$b_k=\frac{2}{\pi\cdot\mu_k}\cdot\frac{\prod_{n=1}^{\bar{n}-1}\left(1-\frac{\mu_k^2}{\sigma^2\cdot\left(\left(\frac{A}{\pi}\right)^2+\left(n-\frac{1}{2}\right)^2\right)}\right)}{\prod_{\substack{n=1\\n\neq k}}^{\bar{n}-1}\left(1-\frac{\mu_k^2}{\mu_n^2}\right)}$$

Inverting the Bessel Transform

Abramowitz & Stegun, p. 484, Eq. 11.3.28

$$\int^{z}\left\{\left(k^2-l^2\right)\cdot t-\frac{\mu^2-\nu^2}{t}\right\}\cdot C_\mu\left(k\cdot t\right)\cdot D_\nu\left(l\cdot t\right)\cdot dt$$

$$=z\cdot\left\{k\cdot C_{\mu+1}\left(k\cdot z\right)\cdot D_\nu\left(l\cdot z\right)-l\cdot C_\mu\left(k\cdot z\right)\cdot D_{\nu+1}\left(l\cdot z\right)\right\}$$

$$-\left(\mu-\nu\right)\cdot C_\mu\left(k\cdot z\right)\cdot D_\nu\left(l\cdot z\right)$$

Our variables (used with order 0 for Taylor, order 1 for Bayliss)

$$k\to\pi\cdot\mu_k,\quad l\to\pi\cdot su,\quad t\to r,\quad z=1,\quad C_0\left(\pi\cdot\mu_k\right)=D_0\left(\pi\cdot\mu_k\right)=J_0\left(\pi\cdot\mu_k\right)$$

Our integral

$$\int_0^1 J_0\left(\pi\cdot\mu_k\cdot r\right)\cdot r\cdot J_0\left(\pi\cdot su\cdot r\right)\cdot dr$$

$$=\frac{\pi\cdot\mu_k\cdot J_1\left(\pi\cdot\mu_k\right)\cdot J_0\left(\pi\cdot su\right)\cdot\mu_k-\pi\cdot su\cdot J_0\left(\pi\cdot\mu_k\right)\cdot J_1\left(\pi\cdot su\right)}{\pi^2\cdot\mu_k^2-\pi^2\cdot su^2}$$

Bessel Transform Pair

$$\int_0^1 J_0\left(\pi\cdot\mu_k\cdot r\right)\cdot r\cdot J_0\left(\pi\cdot su\cdot r\right)\cdot dr = -\frac{J_0\left(\pi\cdot\mu_k\right)}{\pi\cdot\mu_k}\cdot\frac{su}{\mu_k}\cdot\frac{J_1\left(\pi\cdot su\right)}{1-\frac{su^2}{\mu_k^2}}$$

$$\int_0^\infty \frac{su}{\mu_k}\cdot\frac{J_1\left(\pi\cdot su\right)}{1-\frac{su^2}{\mu_k^2}}\cdot su\cdot J_0\left(\pi\cdot su\cdot r\right)\cdot dsu = -\frac{\mu_k}{\pi\cdot J_0\left(\pi\cdot\mu_k\right)}\cdot J_0\left(\pi\cdot\mu_k\cdot r\right)$$

Taylor Weighting Function Result

$$wt(rs)\begin{cases} = 1 + \sum_{k=1}^{\bar{n}-1} at_k \cdot J_0(\pi \cdot \mu_k \cdot rs), rs < 1 \\ = 0, rs > 1 \end{cases}$$

$$at_k = -\frac{\pi \cdot \mu_k}{J_0(\pi \cdot \mu_k)} \cdot b_k = -\frac{1}{J_0(\pi \cdot \mu_k)} \frac{\prod_{n=1}^{\bar{n}-1}\left(1 - \frac{\mu_k^2}{\sigma^2 \cdot \left(\left(\frac{A}{\pi}\right)^2 + \left(n - \frac{1}{2}\right)^2\right)}\right)}{\prod_{\substack{n=1 \\ n \neq k}}^{\bar{n}-1}\left(1 - \frac{\mu_k^2}{\mu_n^2}\right)}$$